DBW-003-1162001 Seat No. ________

M. Sc. (Sem. II) Examination

July - 2022

Mathematics : CMT-2001

(Algebra-II)

Faculty Code : 003
Subject Code : 1162001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) All questions are compulsory.
(2) Each question carries equal marks.
(3) Figure on the right indicate allotted marks.

1 Answer any **seven** short questions : **7×2=14**

(i) Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \in F[X]$ and $a_n \neq 0$. If F contains one root of *f(x)*, then prove that, *f(x)* is reducible over *F*.

(ii) Let $f(x) = x^3 + 6x^2 + 7x + 8$. Prove that, $f(x-2)$ is an irreducible polynomial over *Z[x]*. Is *f(x)* irreducible ? (Y/N).

(iii) $f(x) = x^3 + 4x^2 - 11x + 13$. Prove that $f(x+1)$ is an irreducible polynomial over *Z[x]*.

(iv) Define finite field extension and give an example of finite extension of degree 3.

(v) For the field extension $^{R}|_{Q}$, write down two elements of *R–Q,* which are algebraic over *Q* and write down four elements of *R*, which are not algebraic over *Q* (they are transcendental elements over *Q*).

(vi) Write down the minimal polynomial of the number $\sqrt{2} + \sqrt{3}$ over *Q*.

(vii) Give definition of algebraically closed field. Also give an example of an infinite algebraic extension.

(viii) Give an example of a finite field F such that $|F|=4$.

(ix) Let M be an R-module. In standard notation, prove that, $(-a)m = a(-m) = -(am), \forall\, a \in R \text{ and } \forall\, m \in M$

(x) For a ring R, define R-sub module of an R-module M. Also give an example of an R-sub module.

2 Attempt any two : **2×7=14**

(a) Let p be a prime. Prove that,
$f(x) = x^{p-1} + x^{p-2} + x^{p-3} + \ldots + x + 1 \in \mathbb{Q}[x]$ is an irreducible polynomial over $\mathbb{Q}[x]$.

(b) Let $p(x) \in F[x]$ be an irreducible polynomial. Prove that, there is an extension $^{E}|_{F}$ such that E contains one root of $p(x)$.

(c) Prove that, every finite extension is an algebraic extension.

3 Attempt any one : **1×14=14**

(1) Let $^{E}|_{F}$ be a finite extension. Prove that, following statements are equivalent :

(i) E=F$(\propto)$, for some $\propto \in E$

(ii) There are only a finite number of sub fields of E containing F, as a subfield.

(2) State and prove, the Fundamental Theorem of Galois Theroy.

4 Attempt any **two** : **2×7=14**

(1) Let $^{E}|_{F}$ be a field extension and $\propto_1, \propto_2, \ldots, \propto_n$ be are algebraic over F. Prove that, $^{F(\propto 1, \propto 2, \ldots \propto n}|_{F}$ is a finite field extension.

(2) Let char k = p > 0 and $f(x) \in k[x]$ be an irreducible polynomial. Prove that, $f(x)$ has a multiple root if and only if $f(x) = g(X^p)$, for some g(x) $\in$k[x].

(3) Let K be a field and char K = p>0. Prove that, K is a perfect field if and only if $K = K^P$, where $K^p = \{\alpha^p / \alpha \in K\}$.

5 Attempt any two : **7×2=14**

(a) Let M be a free R-module and $\{e_1, e_2, ...e_n\}$ be a basis for M. Prove that, $M \cong R^n$

(b) Let F be a field and Char F = 0. Let n be a natural number and n[th] root of unity $\in$F. Let $^{K}|_{F}$ be a cyclic extension and [K : F] = n. Prove that, there exist an $\propto \in K$ such that $K = F(\propto)$ and $\propto$ satisfies the polynomial $f(x) = x^n - a \in f[x]$, where $a \in F$.

(c) State and prove, Hilbert Theorem 90.

(d) Let $f : M \to N$ be an R-homomorphism of R-modules. Prove that, Ker *f* and *f(M)* are *R*-sub modules of *M* and *N* respectively.